CHAPTER 2

Complex numbers

Learning objectives

After studying this chapter, you should be able to:

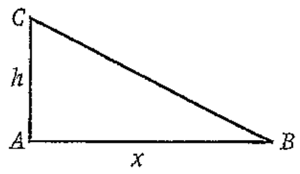
- understand what is meant by a complex number
- find complex roots of quadratic equations
- understand the term complex conjugate
- calculate sums, differences and products of complex numbers
- compare real and imaginary parts of complex numbers

2.1 The historical background to complex numbers

The history of complex numbers goes back to the ancient Greeks who decided, even though they were somewhat puzzled by the fact, that no real number existed that satisfies the equation $x^2 = -1$.

Diophantus, around 275 AD, attempted to solve what seems a reasonable problem, namely:

> Find the sides of a right-angled triangle with perimeter 12 units and area 7 squared units.



Let $AB = x$ and $AC = h$ as shown in the diagram.

The perimeter is $x + h + \sqrt{(x^2 + h^2)}$ and the area is $\frac{1}{2}xh$.

Therefore, $x + h + \sqrt{(x^2 + h^2)} = 12$ and $\frac{1}{2}xh = 7$.

Writing $\sqrt{(x^2 + h^2)} = 12 - (x + h)$ and squaring both sides gives

$$x^2 + h^2 = 12^2 - 24(x + h) + (x^2 + 2hx + h^2)$$

and using $xh = 14$ gives

$$0 = 144 - 24(x + h) + 28$$

or $\quad 0 = 36 - 6(x + h) + 7.$

Multiplying throughout by x gives

$$0 = 36x - 6x^2 - 6hx + 7x$$

or $\quad 0 = 43x - 6x^2 - 84.$

Since $xh = 14$.

which can be rewritten as the quadratic equation

$$6x^2 - 43x + 84 = 0.$$

However, as you learned in C1 chapter 4, when the discriminant $b^2 - 4ac$ of a quadratic expression is negative, the quadratic equation has no real solutions.

Check that $43^2 - 4 \times 6 \times 84$ is negative and therefore the quadratic equation has no real solutions.

Worked example 2.1

The roots of the quadratic equation $x^2 - 3x + 7 = 0$ are α and β.

Without solving the equation, write down the values of $\alpha + \beta$ and $\alpha\beta$ and hence find the value of $\alpha^2 + \beta^2$.

What can you deduce about α and β?

Solution

$\alpha + \beta = 3$ and $\alpha\beta = 7$.

Hence, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9 - 14 = -5$.

It is not possible for the sum of the squares of two real numbers to be negative. Hence, α and β cannot both be real.

You will see later in this chapter that, when this is the situation, we say that α and β are complex numbers.

2.2 The imaginary number i

An imaginary number denoted by the symbol i is introduced so we can represent numbers that are not real. It is defined to have the special property that when you square it you get -1.

If you try to find the square root of -1 on your calculator, it is likely to give you an error message. Try it and see.

> The square root of -1 is not real.
> Because it is imaginary, it is denoted by i where $i^2 = -1$.

This means that when you square 3i you get

$$(3i)^2 = 3^2 \times i^2 = 9 \times -1 = -9.$$

You should now be able to see how we can find the square roots of negative numbers using i.

Worked example 2.2

Simplify each of the following powers of i:

(a) i^3, **(b)** i^4, **(c)** i^7.

Solution

(a) Since $i^3 = i^2 \times i$ and using the fact that $i^2 = -1$, you have $i^3 = -1 \times i = -i$.

(b) $i^4 = i^2 \times i^2 = -1 \times -1 = +1$.
So $i^4 = 1$.

(c) $i^7 = i^4 \times i^3 = +1 \times -i = -i$, using the previous results.
Hence, $i^7 = -i$.

When evaluating powers of i, it is very useful to know that $i^4 = 1$. So, for example

$$i^{29} = (i^4)^7 \times i = 1 \times i = i.$$

> Since $i^2 = -1$, it follows that $i^4 = 1$.
> The powers of i form a periodic cycle of the form $i, -1, -i, +1, \ldots$, so that $i^5 = i$, $i^6 = -1$, etc.

By introducing the symbol i it is possible to solve equations of the form $x^2 = -k$, where k is positive.

Since $(-i)^2 = (-1)^2 \times i^2 = +1 \times -1 = -1$, it follows that:

> The equation $x^2 = -1$ has the two solutions:
> $x = i$ and $x = -i$.

Worked example 2.3

Solve the following equations, giving your answers in terms of i:

(a) $x^2 = -4$, **(b)** $y^2 + 81 = 0$, **(c)** $z^2 = -12$.

Solution

(a) Since the equation $x^2 = -1$ has the two solutions $x = i$ and $x = -i$, and since $4 = 2^2$ it follows that $x^2 = -4$ has solutions $x = 2i$ and $x = -2i$.

(b) Rewriting the equation in the form $y^2 = -81 = -1 \times 9^2$, the two solutions are $y = 9i$ and $y = -9i$.

(c) $z^2 = -12 = -1 \times 12$.
Recall that, using surds, $\sqrt{12} = \sqrt{4} \times \sqrt{3} = 2\sqrt{3}$.
Therefore the equation has solutions $z = 2\sqrt{3}i$ and $z = -2\sqrt{3}i$.

EXERCISE 2A

1 Simplify each of the following:

(a) i^5, **(b)** i^6, **(c)** i^9, **(d)** i^{27},

(e) $(-i)^3$, **(f)** $(-i)^7$, **(g)** $(-i)^{10}$.

2 Simplify:

(a) $(2i)^3$, **(b)** $(3i)^4$, **(c)** $(7i)^2$,

(d) $(-2i)^2$, **(e)** $(-3i)^3$, **(f)** $(-2i)^5$.

3 Solve each of the following equations, giving your answers in terms of i:

(a) $x^2 = -9$, **(b)** $x^2 = -100$, **(c)** $x^2 = -49$,

(d) $x^2 + 1 = 0$, **(e)** $x^2 + 121 = 0$, **(f)** $x^2 + 64 = 0$,

(g) $x^2 + n^2 = 0$, where n is a positive integer.

4 Find the exact solutions of each of the following equations, giving your answers in terms of i:

(a) $x^2 = -5$, **(b)** $x^2 = -3$, **(c)** $x^2 = -8$,

(d) $x^2 + 20 = 0$, **(e)** $x^2 + 18 = 0$, **(f)** $x^2 + 48 = 0$.

5 In 1545, the mathematician Cardano tried to solve the problem of finding two numbers, a and b, whose sum is 10 and whose product is 40.

(a) Show that a must satisfy the equation $a^2 - 10a + 40 = 0$.

(b) What can you deduce about the numbers a and b?

6 A right-angled triangle has area A cm^2 and perimeter P cm. A side other than the hypotenuse has length x cm.
Form a quadratic equation in x in each of the following cases:

(a) $A = 6, P = 12$, **(b)** $A = 3, P = 8$, **(c)** $A = 30, P = 30$.

Hence, for each case above, find the possible values of x whenever real solutions exist.

2.3 Complex numbers and complex conjugates

Whereas a number such as 3i is said to be imaginary, a number such as $3 + 5i$ is said to be complex. It consists of the real number 3 added to the imaginary number 5i.

> A number of the form $p + qi$, where p and q are real numbers and $i^2 = -1$, is called a **complex number**.

Imaginary numbers are in fact a subset of the complex numbers, since any imaginary number can be written as $0 + ki$, where k is real.

The two complex numbers $3 + 5i$ and $3 - 5i$ are said to be **complex conjugates**.

The complex conjugate of $3 + 5i$ is $3 - 5i$ and the complex conjugate of $3 - 5i$ is $3 + 5i$. Consequently, the numbers $3 + 5i$ and $3 - 5i$ are often referred to as a conjugate pair.

You met the idea of *real* conjugates when dealing with surds in C1 where, for example, $2 + \sqrt{3}$ and $2 - \sqrt{3}$ were said to be conjugates.

> The complex conjugate of $p + qi$ is $p - qi$, where p and q are real numbers.
> The conjugate of z is denoted by z^*.

Hence, if $z = -7 + 2i$, you can immediately write down that $z^* = -7 - 2i$.

When $z = 3 - 4i$, for example, $z^* = 3 + 4i$.

Worked example 2.4

Write down the complex conjugates of

(a) $-2 + 3i$, **(b)** $8 - 9i$, and **(c)** $2 + \sqrt{3}i$.

Solution

(a) The complex conjugate of $-2 + 3i$ is $-2 - 3i$; ← Notice that the real part remains as -2.

(b) The complex conjugate of $8 - 9i$ is $8 + 9i$;

(c) The complex conjugate of $2 + \sqrt{3}i$ is $2 - \sqrt{3}i$.

2.4 Combining complex numbers

The operations of addition, subtraction and multiplication using complex numbers are very similar to the way you perform arithmetic with real numbers.

In algebra, you can simplify expressions such as $(3 + 4x) - 2(5 + x)$.

$$(3 + 4x) - 2(5 + x) = 3 + 4x - 10 - 2x = -7 + 2x.$$

Similarly, with complex numbers, you can simplify $(3 + 4i) - 2(5 + i)$.

$$(3 + 4i) - 2(5 + i) = 3 + 4i - 10 - 2i = -7 + 2i.$$

You can multiply complex numbers in the same way that you multiply out brackets in algebra.

$$(4 + 5x)(3 - 2x) = 12 - 8x + 15x - 10x^2 = 12 + 7x - 10x^2.$$

Similarly,

$$\begin{aligned}(4 + 5i)(3 - 2i) &= 12 - 8i + 15i - 10i^2 = 12 + 7i - 10 \times (-1) \\ &= 22 + 7i\end{aligned}$$

Since $i^2 = -1$.

The letter z is often used to represent a complex number. If a question uses more than one complex number, it is common to use subscripts and to write the numbers as z_1, z_2, z_3, etc.

Worked example 2.5

The complex numbers z_1 and z_2 are given by $z_1 = 3 - 5\text{i}$ and $z_2 = -1 + 4\text{i}$.

Find: **(a)** $2z_1 + 5z_2$ **(b)** $3z_1 - 4z_2$
(c) z_1z_2 **(d)** z_1^2

Solution

(a) $2z_1 + 5z_2 = 2(3 - 5\text{i}) + 5(-1 + 4\text{i}) = 6 - 10\text{i} - 5 + 20\text{i}$
$= 1 + 10\text{i}$

(b) $3z_1 - 4z_2 = 3(3 - 5\text{i}) - 4(-1 + 4\text{i}) = 9 - 15\text{i} + 4 - 16\text{i}$
$= 13 - 31\text{i}$

(c) $z_1z_2 = (3 - 5\text{i})(-1 + 4\text{i}) = -3 + 12\text{i} + 5\text{i} - 20\text{i}^2$
$= -3 + 17\text{i} + 20$
$= 17 + 17\text{i}$

(d) $z_1^2 = (3 - 5\text{i})(3 - 5\text{i}) = 9 - 30\text{i} + 25\text{i}^2 = 9 - 30\text{i} - 25$
$= -16 - 30\text{i}$

You will find that when you are solving a quadratic equation with real coefficients, if one of the roots is $3 + 5\text{i}$, for example, then the other root is $3 - 5\text{i}$.

Worked example 2.6

Find the quadratic equation with roots $3 + 5\text{i}$ and $3 - 5\text{i}$.

Solution

The sum of the roots is $(3 + 5\text{i}) + (3 - 5\text{i}) = 6 + 0\text{i} = 6$.

The product of the roots is $(3 + 5\text{i})(3 - 5\text{i})$

$$= 9 + 15\text{i} - 15\text{i} - 5^2\text{i}^2 = 9 - 25 \times (-1) = 9 + 25$$
$$= 34.$$

Hence, the quadratic equation is $x^2 - 6x + 34 = 0$.

> When a quadratic equation with real coefficients has complex roots, these roots are always a pair of complex conjugates.

EXERCISE 2B

1 Find the complex conjugate of each of the following:
(a) $3-\mathrm{i}$, (b) $2+6\mathrm{i}$, (c) $-3-8\mathrm{i}$, (d) $-7+5\mathrm{i}$,
(e) $3+\sqrt{2}\mathrm{i}$, (f) $4-\sqrt{3}\mathrm{i}$, (g) $-1-\frac{1}{3}\mathrm{i}$

2 Simplify each of the following:
(a) $(3+\mathrm{i})+(5-2\mathrm{i})$, (b) $(3+\mathrm{i})-(5-2\mathrm{i})$,
(c) $3(3-5\mathrm{i})+4(1+6\mathrm{i})$, (d) $3(3-5\mathrm{i})-4(1+6\mathrm{i})$,
(e) $4(8-5\mathrm{i})-5(1-4\mathrm{i})$, (f) $6(3-4\mathrm{i})-2(9-6\mathrm{i})$.

3 Simplify each of the following:
(a) $(4+\mathrm{i})(7-2\mathrm{i})$, (b) $(3+4\mathrm{i})(5-3\mathrm{i})$,
(c) $(7-5\mathrm{i})(5+6\mathrm{i})$, (d) $3(3-5\mathrm{i})(4+3\mathrm{i})$,
(e) $(8-5\mathrm{i})(8-5\mathrm{i})$, (f) $(3-4\mathrm{i})^2$.

4 Find the square of each of the following complex numbers:
(a) $3-\mathrm{i}$, (b) $2+6\mathrm{i}$,
(c) $-3-8\mathrm{i}$, (d) $-7+5\mathrm{i}$,
(e) $3+\sqrt{2}\mathrm{i}$, (f) $4-\sqrt{3}\mathrm{i}$.

5 The complex numbers z_1 and z_2 are given by $z_1=2-3\mathrm{i}$ and $z_2=-3+5\mathrm{i}$.
Find: (a) $5z_1+3z_2$ (b) $3z_1-4z_2$ (c) z_1z_2

6 Given that z^* is the conjugate of z, find the values of
(i) $z+z^*$, (ii) $z-z^*$, (iii) zz^*
for each of the following values of z:
(a) $2+3\mathrm{i}$, (b) $-4+2\mathrm{i}$, (c) $-5-3\mathrm{i}$, (d) $6-5\mathrm{i}$,
(e) $x+y\mathrm{i}$, where x and y are real.

7 Find the value of the real constant p so that $(3+2\mathrm{i})(4-\mathrm{i})+p$ is purely imaginary.

8 Find the value of the real constant q so that $(2+5\mathrm{i})(4-3\mathrm{i})+q\mathrm{i}$ is real.

9 (a) Find $(1+\mathrm{i})(2-3\mathrm{i})$.
(b) Hence, simplify $(1+\mathrm{i})(2-3\mathrm{i})(5+\mathrm{i})$.

10 (a) Find $(2+\mathrm{i})^2$.
(b) Hence, find: (i) $(2+\mathrm{i})^3$, (ii) $(2+\mathrm{i})^4$.

2.5 Complex roots of quadratic equations

Recall that the quadratic equation $ax^2+bx+c=0$ will not have real roots when $b^2-4ac<0$.

Perhaps the easiest way to solve quadratic equations with complex roots is to complete the square.

2

Worked example 2.7

Solve the following quadratic equations by completing the square.

(a) $x^2 - 4x + 13 = 0$,

(b) $x^2 + 6x + 25 = 0$,

(c) $x^2 - 2x + 7 = 0$.

Solution

(a) $x^2 - 4x + 13 = (x-2)^2 + 9$

$$x^2 - 4x + 13 = 0 \Rightarrow (x-2)^2 + 9 = 0 \Rightarrow (x-2)^2 = -9$$
$$\Rightarrow (x-2) = \pm 3\text{i} \Rightarrow x = 2 \pm 3\text{i}$$

Hence, the roots are $2 + 3\text{i}$ and $2 - 3\text{i}$.

Notice that the solutions are complex conjugates.

(b) $x^2 + 6x + 25 = (x+3)^2 + 16$

$$x^2 + 6x + 25 = 0 \Rightarrow (x+3)^2 + 16 = 0 \Rightarrow (x+3)^2 = -16$$
$$\Rightarrow (x+3) = \pm 4\text{i} \Rightarrow x = -3 \pm 4\text{i}$$

Hence, the roots are $-3 + 4\text{i}$ and $-3 - 4\text{i}$.

You can check your answers by finding the sum of the roots and the product of the roots.

(c) $x^2 - 2x + 7 = (x-1)^2 + 6$

$$x^2 - 2x + 7 = 0 \Rightarrow (x-1)^2 + 6 = 0 \Rightarrow (x-1)^2 = -6$$
$$\Rightarrow (x-1) = \pm\sqrt{6}\text{i} \Rightarrow x = 1 \pm \sqrt{6}\text{i}$$

Hence, the roots are $1 + \sqrt{6}\text{i}$ and $1 - \sqrt{6}\text{i}$.

Worked example 2.8

Solve the following quadratic equations by using the quadratic equation formula.

(a) $x^2 - 10x + 26 = 0$, (b) $2x^2 + 4x + 5 = 0$.

An alternative method is to use the quadratic equation formula, but this is sometimes more difficult than completing the square.

Solution

(a) Using $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1$, $b = -10$, $c = 26$:

$$x = \frac{10 \pm \sqrt{100 - 104}}{2}$$

Therefore, $x = \dfrac{10 \pm 2\text{i}}{2} = 5 \pm \text{i}$.

The solutions are $x = 5 + \text{i}$ and $x = 5 - \text{i}$.

(b) Using $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 2$, $b = 4$, $c = 5$:

$$x = \frac{-4 \pm \sqrt{16 - 40}}{4}$$

Therefore, $x = \dfrac{-4 \pm \sqrt{24}\text{i}}{4} = \dfrac{-4 \pm 2\sqrt{6}\text{i}}{4} = -1 \pm \dfrac{\sqrt{6}}{2}\text{i}$.

The solutions are $x = -1 + \dfrac{\sqrt{6}}{2}\text{i}$ and $x = -1 - \dfrac{\sqrt{6}}{2}\text{i}$.

EXERCISE 2C

Find the complex roots of each of the following equations:

1 $x^2 - 2x + 5 = 0$	2 $x^2 + 4x + 13 = 0$
3 $x^2 - 2x + 10 = 0$	4 $x^2 - 6x + 25 = 0$
5 $x^2 - 8x + 20 = 0$	6 $x^2 + 4x + 5 = 0$
7 $x^2 - 12x + 40 = 0$	8 $x^2 + 2x + 50 = 0$
9 $x^2 + 8x + 17 = 0$	10 $x^2 - 10x + 34 = 0$
11 $2x^2 - 2x + 5 = 0$	12 $9x^2 + 6x + 10 = 0$
13 $4x^2 - 8x + 5 = 0$	14 $5x^2 - 6x + 5 = 0$
15 $13x^2 + 10x + 13 = 0$	16 $x^2 - 2x + 4 = 0$
17 $x^2 + 4x + 9 = 0$	18 $x^2 - 6x + 16 = 0$
19 $x^2 + 8x + 19 = 0$	20 $x^2 - x + 1 = 0$

2.6 Equating real and imaginary parts

The complex number $3 + 4i$ is said to have

real part equal to 3

and imaginary part equal to 4.

A common mistake is to say that the imaginary part is 4i, when it should be simply 4.

In general, when $z = p + qi$, where p and q are real numbers,

- the real part of z is p, and
- the imaginary part of z is q.

When an equation involves complex numbers, it is in fact TWO equations since the real parts must be the same on both sides of the equation and the imaginary parts must be equal as well.

Worked example 2.9

Find the value of each of the real constants a and b such that $(a + 2i)^2 = b + 12i$.

Solution

The left-hand side can be multiplied out to give

$$a^2 + 4ai + 4i^2 = a^2 - 4 + 4ai.$$

Hence, $a^2 - 4 + 4ai = b + 12i$.

Equating real parts gives $a^2 - 4 = b$ [1]

and equating imaginary parts gives $4a = 12$ [2]

From [2], $a = 3$.

Substituting into [1] gives $9 - 4 = b$, so $b = 5$.

Worked example 2.10

Find the complex number z which satisfies the equation $2z + 3z^* = 10 + 4i$.

Solution

Let $z = x + yi$, where x and y are real.

Therefore, the conjugate $z^* = x - yi$.

$$2z + 3z^* = 2(x + yi) + 3(x - yi) = 2x + 2yi + 3x - 3yi = 5x - yi$$

Hence, $5x - yi = 10 + 4i$.

Equating real parts gives $5x = 10 \Rightarrow x = 2$.

Equating imaginary parts gives $-yi = 4i \Rightarrow y = -4$.

Hence, z is the complex number $2 - 4i$.

Worked example 2.11

The complex number $1 + 3i$ is a root of the quadratic equation $z^2 - (2 + 5i)z + p + iq = 0$, where p and q are real.

(a) Find the values of p and q.

(b) By considering the sum of the roots, find the second root of the quadratic equation. Why are the two roots not complex conjugates?

Solution

(a) Substituting $z = 1 + 3i$ into $z^2 - (2 + 5i)z + p + iq = 0$ gives

$$(1 + 3i)^2 - (2 + 5i)(1 + 3i) + p + iq = 0.$$

Expanding gives $1 + 6i - 9 - (2 + 5i + 6i - 15) + p + iq = 0$

or $\quad -8 + 6i + 13 - 11i + p + iq = 0.$

Equating real parts gives $p = -5$.

Equating imaginary parts gives $q = 5$.

(b) The sum of the roots is $2 + 5i$ and one root is known to be $1 + 3i$. The other root must be $1 + 2i$.

The roots are not conjugates because the quadratic equation did not have real coefficients.

An interesting check can now be made of the values of p and q.

The product of the roots is $(1 + 3i)(1 + 2i) = 1 + 5i - 6 = -5 + 5i$, which confirms that the values of p and q were correct.

Worked example 2.12

Find the square roots of the complex number $-5 + 12i$.

Solution

You need to find possible complex numbers z such that $z^2 = -5 + 12i$.

Let $z = x + yi$, where x and y are real.

$$z^2 = (x + yi)^2 = x^2 + 2xyi - y^2 = -5 + 12i.$$

Equating real and imaginary parts gives

$$x^2 - y^2 = -5 \qquad [1]$$

and $2xy = 12$ or $xy = 6$ [2]

Substitute $y = \frac{6}{x}$ into [1]

$$x^2 - \frac{36}{x^2} = -5 \Rightarrow x^4 - 36 = -5x^2$$

or $x^4 + 5x^2 - 36 = 0 \Rightarrow (x^2 + 9)(x^2 - 4) = 0$

So either $x^2 = -9$ or $x^2 = 4$.

But since x is real, the only possibility is that $x^2 = 4$. This gives two values for x, $x = 2$ or $x = -2$.

Since $y = \frac{6}{x}$, $x = 2 \Rightarrow y = 3$ and $x = -2 \Rightarrow y = -3$.

Hence, the two square roots of $-5 + 12i$ are $2 + 3i$ and $-2 - 3i$.

EXERCISE 2D

1 Find the value of each of the real constants a and b such that $(a + 3i)^2 = 8b + 30i$.

2 Find the value of each of the real constants p and q such that $(3 + 4i)(p + 2i) = q + 26i$.

3 Find the value of each of the real constants t and u such that $(2 + 2i)^2(t + 3i) = u + 32i$.

4 Find the complex number z so that $3z + 4z^* = 28 + i$.

5 Find the complex number z so that $z + 3z^* = 12 - 8i$.

6 Find the complex number z so that $z - 4iz^* + 2 + 7i = 0$.

7 Find all possible values of z such that $z + z^* = 6$.

8 Find the square roots of each of the following complex numbers:

(a) $-7 + 24\text{i}$, **(b)** $35 - 12\text{i}$, **(c)** $21 + 20\text{i}$.

9 The complex number $2 + 5\text{i}$ is a root of the quadratic equation $z^2 - (3 + 7\text{i})z + p + \text{i}q = 0$, where p and q are real.

(a) Find the values of p and q.

(b) By considering the sum of the roots, find the second root of the quadratic equation. Why are the two roots not complex conjugates?

10 **(a)** Find the square roots of $-3 + 4\text{i}$.

(b) Hence, or otherwise, find the two complex roots of the quadratic equation $z^2 + 2(1 + \text{i})z + 3 - 2\text{i} = 0$.

Key point summary

1 The square root of -1 is not real. Because it is imaginary, it is denoted by i where $\text{i}^2 = -1$. *p17*

2 Since $\text{i}^2 = -1$, it follows that $\text{i}^4 = 1$. *p18*
The powers of i form a periodic cycle of the form i, -1, $-\text{i}$, $+1$, …, so that $\text{i}^5 = \text{i}$, $\text{i}^6 = -1$, etc.

3 The equation $x^2 = -1$ has the two solutions: $x = \text{i}$ and $x = -\text{i}$. *p18*

4 A number of the form $p + q\text{i}$, where p and q are real numbers and $\text{i}^2 = -1$, is called a complex number. *p19*

5 The complex conjugate of $p + q\text{i}$ is $p - q\text{i}$, where p and q are real numbers. *p20*
The conjugate of z is denoted by z^*.

6 When a quadratic equation with real coefficients has complex roots, these roots are always a pair of complex conjugates. *p21*

7 In general, when $z = p + q\text{i}$, where p and q are real numbers, *p24*

- the real part of z is p, and
- the imaginary part of z is q.

Answers

EXERCISE 1D

1 $k = 20$. 2 $k = \pm 8$. 3 $7x^2 - 9x - 9 = 0$.

4 $2x^2 + 13x + 17 = 0$. 5 $x^2 - 16x + 40 = 0$.

6 (a) (i) 39, (ii) -230; (b) $7x^2 - 230x - 49 = 0$.

7 (a) (i) $\frac{22}{9}$, (ii) $-\frac{22}{27}$; (b) $81x^2 + 66x + 1 = 0$.

8 (a) (i) $\alpha + \beta = -2$, $\alpha\beta = 3$, (iii) $\frac{10}{27}$;
(b) $27x^2 - 10x + 1 = 0$.

9 (a) (ii) 18; (b) (ii) 47; (c) $x^2 - 15x - 45 = 0$.

10 (a) $\alpha + \beta = -3$, $\alpha\beta = -2$;
(b) (i) $\frac{13}{4}$, (ii) $-\frac{17}{4}$; (c) $4x^2 + 51x - 17 = 0$.

2 Complex numbers

EXERCISE 2A

1 (a) i; (b) -1; (c) i; (d) $-$i;
(e) i; (f) i; (g) -1.

2 (a) -8i; (b) 81; (c) -49;
(d) -4; (e) 27i; (f) -32i.

3 (a) -3i, 3i; (b) -10i, 10i; (c) -7i, 7i; (d) $-$i, i;
(e) -11i, 11i; (f) -8i, 8i; (g) $-n$i, ni.

4 (a) $-\sqrt{5}$i, $\sqrt{5}$i; (b) $-\sqrt{3}$i, $\sqrt{3}$i; (c) $-2\sqrt{2}$i, $2\sqrt{2}$i
(d) $-2\sqrt{5}$i, $2\sqrt{5}$i; (e) $-3\sqrt{2}$i, $3\sqrt{2}$i; (f) $-4\sqrt{3}$i, $4\sqrt{3}$i.

5 (b) Not real.

6 (a) $x^2 - 7x + 12 = 0$, $x = 3, 4$;
(b) $4x^2 - 19x + 24 = 0$, no real solutions;
(c) $x^2 - 17x + 60 = 0$, $x = 5, 12$.

EXERCISE 2B

1 (a) $3 + $i; (b) $2 - 6$i; (c) $-3 + 8$i; (d) $-7 - 5$i;
(e) $3 - \sqrt{2}$i; (f) $4 + \sqrt{3}$i; (g) $-1 + \frac{1}{3}$i.

2 (a) $8 - $i; (b) $-2 + 3$i; (c) $13 + 9$i;
(d) $5 - 39$i; (e) 27; (f) -12i.

3 (a) $30 - $i; (b) $27 + 11$i; (c) $65 + 17$i;
(d) $81 - 33$i; (e) $39 - 80$i; (f) $-7 - 24$i.

4 (a) $8 - 6$i; (b) $-32 + 24$i; (c) $-55 + 48$i;
(d) $24 - 70$i; (e) $7 + 6\sqrt{2}$i; (f) $13 - 8\sqrt{3}$i.

5 (a) 1; (b) $18 - 29$i; (c) $9 + 19$i.

6 (a) (i) 4, (ii) 6i, (iii) 13;
(b) (i) -8, (ii) 4i, (iii) 20;
(c) (i) -10, (ii) -6i, (iii) 34;
(d) (i) 12, (ii) -10i, (iii) 61;
(e) (i) $2x$, (ii) $2y$i, (iii) $x^2 + y^2$.

7 $p = -14$. 8 $q = -14$.

9 (a) $5 - $i; (b) 26.

10 (a) $3 + 4$i; (b) (i) $2 + 11$i, (ii) $-7 + 24$i.

EXERCISE 2C

1 $1 \pm 2i$. **2** $-2 \pm 3i$. **3** $1 \pm 3i$. **4** $3 \pm 4i$.

5 $4 \pm 2i$. **6** $-2 \pm i$. **7** $6 \pm 2i$. **8** $-1 \pm 7i$.

9 $-4 \pm i$. **10** $5 \pm 3i$. **11** $\frac{1}{2} \pm \frac{3}{2}i$. **12** $-\frac{1}{3} \pm i$.

13 $1 \pm \frac{1}{2}i$. **14** $\frac{3}{5} \pm \frac{4}{5}i$. **15** $-\frac{5}{13} \pm \frac{12}{13}i$. **16** $1 \pm \sqrt{3}i$.

17 $-2 \pm \sqrt{5}i$. **18** $3 \pm \sqrt{7}i$. **19** $-4 \pm \sqrt{3}i$. **20** $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$.

EXERCISE 2D

1 $a = 5, b = 2$. **2** $p = 5, q = 7$. **3** $t = 4, u = -24$.

4 $z = 4 - i$. **5** $z = 3 + 4i$. **6** $z = 2 + i$.

7 $3 + iy$, where y is real.

8 **(a)** $3 + 4i, -3 - 4i$; **(b)** $-6 + i, 6 - i$; **(c)** $5 + 2i, -5 - 2i$.

9 **(a)** $p = -8, q = 9$; **(b)** $1 + 2i$, coefficients are not real.

10 **(a)** $1 + 2i, -1 - 2i$; **(b)** $i, -2 - 3i$.

3 Inequalities

EXERCISE 3A

1 $x < 2, x > \frac{5}{2}$. **2** $4 < x < 7$.

3 $x < 2, x \geqslant \frac{11}{2}$. **4** $\frac{13}{2} < x < 7$.

5 $-4 < x < -\frac{7}{3}$. **6** $6 < x < \frac{29}{4}$.

7 $-4 < x < \frac{13}{5}$. **8** $0 \leqslant x < 3, x \geqslant 4$.

9 $\frac{5}{2} < x \leqslant 3, x \geqslant 5$. **10** $-8 < x < -1$ and $x > 4$.

EXERCISE 3B

1 $x < -6, x > \frac{5}{2}$. **2** $-\frac{13}{2} < x < 3$.

3 $-2 < x \leqslant 1, x \geqslant 2$. **4** $-2 < x < 2, x > 3$.

5 $-8 < x < -2, x > 1$. **6** $-4 < x < -2, -1 < x < 2$.

7 $x < -3, -2 < x < 0, x > 6$. **8** $0 \leqslant x < 3, x \leqslant -2$.

9 $-4 < x \leqslant -\frac{11}{9}, x > 1$. **10** $\frac{11}{5} < x < 3, x > 5$.

EXERCISE 3C

1 $-2 < x \leqslant 5$. **2** $x < -3, -1 < x \leqslant 0$.

3 $x \leqslant -\frac{9}{2}, x > 1$. **4** $0 < x < 4$.

5 $x < -6, -5 < x < 1$. **6** $\frac{9}{4} \leqslant x \leqslant 4, x \neq 3$.

7 $x < -\frac{2}{3}, 0 < x < 2, x > 3$. **8** $x < -5, -2 < x < 19$.

9 $2 < x < 4, -3 < x < 1$. **10** $x < -6, -5 < x < -3$.

11 $0 < x < 1, x > 2$. **12** $x < 2, x \geqslant 3$.

13 **(b)** Step 1; **(c)** $2 < x \leqslant 3$