AQA Further Pure 3 Numerical methods

Section 1: Euler's method

Notes and Examples

These notes contain subsections on

Terminology and notation

In this section you will be looking at step-by-step methods to find numerical solutions of first order differential equations. In FP1 you found numerical solutions of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)$ by using a step-by-step method based on the linear approximation $y_{n+1} \approx y_n + h\,\mathrm{f}\left(x_n\right)$, where the step length is h, so $x_{n+1} = x_n + h$.
In this chapter the work is extended to solving differential equations of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$.

You will be given a differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$.
The solution of the differential equation is denoted by $y(x)$.
You will be given a boundary condition, normally in the form $y\left(x_0\right) = y_0$, where x_0 and y_0 are numerical. This is the same as the boundary condition $y = y_0$ when $x = x_0$.
As before, h is such that $x_{n+1} = x_n + h$ and is called the step length. In this section you may not always be given the value of h.
In general $y\left(x_r\right) = y_r$, where $x_r = x_0 + rh$.

It is extremely important that you understand the notation used. The following example uses some of the notation.

Example 1
The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 + y^2$$

is to be solved subject to the boundary condition $y(0.5) = 1$.

The values of $y(0.55)$, $y(0.6)$, … , $y(1)$ are required. In this case:

(i) state the expression for $\mathrm{f}(x, y)$;
(ii) state the values of x_0, y_0, h, x_1 and x_2;
(iii) state the value of n such that y_n corresponds to $y(1)$;
(iv) evaluate $\mathrm{f}(x_0, y_0)$.

Solution

(i) Comparing $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$ with $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 + y^2$
gives $\mathrm{f}(x, y) = 4x^3 + y^2$.

(ii) Comparing the boundary condition $y(0.5) = 1$ with $y(x_0) = y_0$
gives $x_0 = 0.5$ and $y_0 = 1$.

The step length, $h = 0.55 - 0.5 = 0.05$

Using $x_r = x_0 + rh$
gives $x_1 = 0.55$ and $x_2 = 0.6$

(iii) $y(x_r) = y_r$, where $x_r = x_0 + rh$.
$y(1) = y_n$, so $1 = 0.5 + n(0.05)$
$\Rightarrow n = 10$

(iv) Using
$$\begin{aligned}\mathrm{f}(x_0, y_0) &= 4x_0^3 + y_0^2 \\ &= 4(0.5)^3 + 1^2 \\ &= 1.5\end{aligned}$$

Euler's formula

Euler's formula, with the standard notation, for solving the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$ is

$$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r)$$

This formula appears in the AQA booklet of formulae.

Example 2

The function $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $f(x, y) = \sin(x^2 + y)$

and $y(1) = 1.5$

Show that the Euler formula

$$y_{r+1} = y_r + h\,f(x_r, y_r)$$

with $h = 0.1$, gives $y(1.1) = 1.560$ correct to three decimal places.

Solution

In this example, $x_0 = 1$, $y_0 = 1.5$, $h = 0.1$, $f(x, y) = \sin(x^2 + y)$ and $y(1.1) = y_1$

Using $\quad y_{r+1} = y_r + h\,f(x_r, y_r)$

with $r = 0$ gives $\quad y_1 = y_0 + h\sin({x_0}^2 + y_0)$

$$= 1.5 + 0.1\sin(1^2 + 1.5)$$

$$= 1.5 + 0.1\sin 2.5$$

$$= 1.5598...$$

$y(1.1) = 1.560$ correct to three decimal places.

Calculator must be set in radian mode.

A value to at least 4 decimal places must be shown to justify being able to write the printed 3 decimal places answer

The mid-point formula

The mid-point formula, with the standard notation, for solving the differential equation $\frac{dy}{dx} = f(x, y)$ is

$$y_{r+1} = y_{r-1} + 2h\,f(x_r, y_r)$$

If this formula is required in the examination it will always be stated in the question. You do not need to learn it for the MFP3 exam.

You can only use the mid-point formula if you know the previous two values of y. In examination questions this topic is usually combined with Euler's formula. The next example illustrates this.

Example 3

The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 + \frac{\sqrt{y}}{2}$$

and $y(0) = 1$.

(i) Use the Euler formula

$$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r)$$

with $h = 0.5$, to find an approximation to $y(0.5)$.

(ii) Use the formula

$$y_{r+1} = y_{r-1} + 2h\,\mathrm{f}(x_r, y_r)$$

to find an approximation to the value of $y(1)$, giving your answer to three decimal places.

Solution

(i) In this example, $x_0 = 0$, $y_0 = 1$, $h = 0.5$, $\mathrm{f}(x, y) = x^3 + \frac{\sqrt{y}}{2}$ and $y(0.5) = y_1$

Using $$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r)$$

with $r = 0$ gives
$$\begin{aligned} y_1 &= y_0 + h\left(x_0^{\,3} + \frac{\sqrt{y_0}}{2}\right) \\ &= 1 + 0.5 \times \left(0 + \frac{1}{2}\right) \\ &= 1.25 \end{aligned}$$

So $y(0.5) = 1.25$

(ii) In this example, $x_0 = 0$, $y_0 = 1$, $h = 0.5$, $x_1 = 0.5$, $y_1 = 1.25$ and $y(1) = y_2$

Using $$y_{r+1} = y_{r-1} + 2h\,\mathrm{f}(x_r, y_r)$$

with $r = 1$ gives
$$\begin{aligned} y_2 &= y_0 + 2 \times 0.5 \times \left(x_1^{\,3} + \frac{\sqrt{y_1}}{2}\right) \\ &= 1 + \left(0.5^3 + \frac{\sqrt{1.25}}{2}\right) \\ &= 1.68401 \end{aligned}$$

$y(1) = 1.684$ to 3 d.p.

The improved Euler formula

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The improved Euler formula, with the standard notation, for solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$ is

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\,\mathrm{f}(x_r, y_r)$

and $k_2 = h\,\mathrm{f}(x_r + h, y_r + k_1)$

As before, if this formula is required in the examination it will always be stated in the question. You do not need to learn it for the MFP3 exam.

Some students prefer to show the various values in a table. In the exam, if you use this approach, you should always show the examiner how you obtained the various values.

Example 4

The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $\mathrm{f}(x, y) = 2y + 3 - 12x^3$
and $y(0) = 2$

(i) Use the improved Euler formula

$$y_{r+1} = y_r + \tfrac{1}{2}(k_1 + k_2)$$

where $k_1 = h\,\mathrm{f}(x_r, y_r)$

and $k_2 = h\,\mathrm{f}(x_r + h, y_r + k_1)$

with $h = 0.25$ to obtain an approximation to $y(0.25)$

(ii) Use the formula given in part (i), together with your value for $y(0.25)$ obtained in part (i), to obtain an approximation to $y(0.5)$, giving your answer to four decimal places.

Solution

(i) $x_0 = 0$, $y_0 = 2$, $h = 0.25$ and $y(0.25) = y_1$

r	x_r	y_r	k_1	$x_r + h$	$y_r + k_1$	k_2	y_{r+1}
0	0	2	1.75	0.25	3.75	2.578125	4.1640625

$$\mathrm{f}(x, y) = 2y + 3 - 12x^3$$

$$k_1 = h\,\mathrm{f}\left(x_r,\ y_r\right) = 0.25\ \mathrm{f}(0,\ 2)$$
$$= 0.25(2\times 2+3-12\times 0^3)$$
$$= 1.75$$

$$k_2 = h\,\mathrm{f}\left(x_r+h,\ y_r+k_1\right) = 0.25\ \mathrm{f}(0.25,\ 3.75)$$
$$= 0.25(2\times 3.75+3-12\times 0.25^3)$$
$$= 2.578125$$

$$y_1 = y_0 + \tfrac{1}{2}\left(k_1+k_2\right)$$
$$= 2+\tfrac{1}{2}\left(1.75+2.578125\right)$$
$$= 4.1640625$$

(ii)

r	x_r	y_r	k_1	x_r+h	y_r+k_1	k_2	y_{r+1}
0	0	2	1.75	0.25	3.75	2.578125	4.1640625
1	0.25	4.1640625	2.78515625	0.5	6.94921875	3.849609375	7.481445313

$x_1 = 0$, $y_1 = 4.1640625$, $h = 0.25$ and $y(0.5) = y_2$

$\mathrm{f}(x, y) = 2y + 3 - 12x^3$

$$k_1 = h\,\mathrm{f}\left(x_r,\ y_r\right) = 0.25\ \mathrm{f}(0.25,\ 4.1640625)$$
$$= 0.25(2\times 4.1640625+3-12\times 0.25^3)$$
$$= 2.78515625$$

$$k_2 = h\,\mathrm{f}\left(x_r+h,\ y_r+k_1\right) = 0.25\ \mathrm{f}(0.5,\ 6.94921875)$$
$$= 0.25(2\times 6.94921875+3-12\times 0.5^3$$
$$= 3.849609375$$

$$y_2 = y_1 + \tfrac{1}{2}\left(k_1+k_2\right)$$
$$= 4.1640625+\tfrac{1}{2}\left(2.78515625+3.849609375\right)$$
$$= 7.481445313$$

$y(0.5) = 7.4814$ (to 4 d.p.)

The Geogebra resource ***Numerical solution of differential equations*** shows Euler's formula, the mid-point formula and Euler's improved formula.

Error analysis: some practical considerations

Students will not be asked, in the examination, to derive the formulae used in this chapter.
Students may be asked to compare the results of applying different formulae as illustrated in the next example.

Example 5
The function $y(x)$ satisfies the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x, y)$$

where $\mathrm{f}(x, y) = x \ln x + \dfrac{y}{x}$

and $y(1) = 1$

(a) (i) Use the Euler formula

$$y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r)$$

with $h = 0.1$, to find an approximation to $y(1.1)$.

(ii) Use this Euler formula again with $h = 0.1$, to find an approximation to $y(1.2)$, giving your answer to four decimal places.

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2h\,\mathrm{f}(x_r, y_r)$$

with your answer to part (a)(i) to obtain an approximation to $y(1.2)$, giving your answer to four decimal places.

(c) Verify that the exact solution of the differential equation is $y = x^2 \ln x - x^2 + 2x$.

(d) Hence calculate, to two decimal places, the percentage errors in the values of $y(1.2)$ obtained in parts (a)(ii) and (b) and comment on your answers.

Solution

(a) (i) $x_0 = 1,\ y_0 = 1,\ h = 0.1,\ \mathrm{f}(x, y) = x \ln x + \dfrac{y}{x}$ and $y(1.1) = y_1$

Using $y_{r+1} = y_r + h\,\mathrm{f}(x_r, y_r)$

with $r = 0$ gives $y_1 = y_0 + h\left[x_0 \ln x_0 + \dfrac{y_0}{x_0}\right]$

$$= 1 + 0.1[0 + 1]$$
$$= 1.1$$

(ii) $x_1 = 1.1$, $y_1 = 1.1$, $h = 0.1$, $\mathrm{f}(x, y) = x\ln x + \dfrac{y}{x}$ and $y(1.2) = y_2$

Using $y_2 = y_1 + h\left[x_1 \ln x_1 + \dfrac{y_1}{x_1}\right]$

$$= 1.1 + 0.1[1.1\ln 1.1 + 1]$$
$$= 1.21048...$$

$y(1.2) = 1.2105$ (to 4 d.p.)

(b) In this case, $x_0 = 1$, $y_0 = 1$, $h = 0.1$, $x_1 = 1.1$, $y_1 = 1.1$ and $y(1.2) = y_2$

Using $y_{r+1} = y_{r-1} + 2h\,\mathrm{f}(x_r, y_r)$

with $r = 1$ gives $y_2 = y_0 + 2\times 0.1\times\left(x_1 \ln x_1 + \dfrac{y_1}{x_1}\right)$

$$= 1 + 0.2\times(1.1\ln 1.1 + 1)$$
$$= 1.22096...$$

$y(1.2) = 1.2210$ (to 4 d.p.)

(c) $y = x^2 \ln x - x^2 + 2x$

$$\frac{dy}{dx} = (2x\ln x + x) - 2x + 2 = 2x\ln x - x + 2$$

$$x\ln x + \frac{y}{x} = x\ln x + (x\ln x - x + 2) = 2x\ln x - x + 2$$

$$\Rightarrow \frac{dy}{dx} = x\ln x + \frac{y}{x}$$

When $x = 1$, $y = \ln 1 - 1 + 2 = 1$

So $y = x^2 \ln x - x^2 + 2x$ is the solution of the differential equation such that $y(1) = 1$.

(d) Using the exact solution, $y(1.2) = 1.2^2\ln 1.2 - 1.2^2 + 2.4 = 1.222543\ldots$

Percentage error using (a)(ii) $= \dfrac{1.222543... - 1.2105}{1.222543...}\times 100$

$$= 0.98508121...$$
$$= 0.99 \text{ (to 2 d.p.)}$$

Percentage error using (b) $= \dfrac{1.222543... - 1.2210}{1.222543...}\times 100$

$$= 0.126215$$
$$= 0.13 \text{ (to 2 d.p.)}$$

Although both formulae have given the solution to a good degree of accuracy, use of the mid-point formula in (b) is significantly more accurate than the use of Euler's formula used in (a).